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PARTITION OF GRAPHS WITH CONDITION ON THE CONNECTIVITY AND MINIMUM DEGREE

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C. Thomassen and M. Szegedy proved the existence of a function $f(s, t)$ such that the points of any $f(s, t)$-connected graph have a decomposition into two non-empty sets such that the subgraphs induced by them are s-connected and t-connected, respectively. We prove, that $f(s, t) \leqq 4s+4t-13$ and examine a similar problem for the minimum degree.

1. Introduction

E. Győri suggested the following problem in the Sixth Hungarian Colloqium on Combinatorics held at Eger, 1981: given integers $s, t \geqq 3$, does there exist a number $f(s, t)$ such that every $f(s, t)$-connected graph $V(G)$ admits a proper partition $\{S, T\}$ of the vertex-set $V(G)$ so that the induced subgraphs $G(S)$ and $G(T)$ are s-connected and t-connected, respectively. Thomassen [1], and independently M. Szegedy established the existence of the function $f(s, t)$. In his proof Thomassen showed the existence of a function $g(s, t)$ (where s, t are natural numbers) such that the vertex-set of any graph G with minimum degree at least $g(s, t)$ has a decomposition $S \cup T$ such that $G(S)$ and $G(T)$ have minimum degree s and t, respectively. Let $f(s, t)$ and $g(s, t)$ be minimal numbers under the above conditions. In his paper Thomassen gave rather weak bounds. We sharpen the estimates of the functions $f(s, t)$ and $g(s, t)$.

2. Notation

Let G be a simple graph, $V(G)$ is the set of points of G, $E(G)$ is the set of edges of G, $S \subseteq V(G)$ and $x \in V(G)$. We use further the following notation. $G(S)$ is the subgraph of G induced by S, $e(S)$ the number of edges of $G(S)$, $d(S)$ the minimum degree in $G(S)$, $d(x)$ the degree of point x, $d(x, S)$ the number of edges (x, y), where $y \in S$. $\{S, T\}$ is a partition of the set $V(G)$ if S and T are non-empty, disjoint subsets of $V(G)$ such that $S \cup T = V(G)$.

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3. Estimating the function $g(s, t)$

3.1. Theorem. *If* $s \geqq 4$, *then* $g(s, t) \leqq t+2s-3$.

Proof. We have to prove that is $d(V(G)) \geqq t+2s-3$, then there exists a partition $\{S, T\}$ of the set $V(G)$ such that $d(S) \geqq s$ and $d(T) \geqq t$.

Let S be a vertex-set such that $|S|$ is minimal and (if there are more than one such S) $e(S)$ is maximal under the conditions:

$$(*) \qquad \begin{cases} e(S) > (s-1)|S| - \dfrac{s(s-1)}{2} \\ |S| \geqq s. \end{cases}$$

Such an S exists (for example $V(G)$ satisfies condition $(*)$). $S \neq V(G)$ as for any $x \in V(G)$,

$$d(V(G) \setminus \{x\}) \geqq t+2s-4,$$

so if $t \geqq 2$, then

$$e(V(G) \setminus \{x\}) \geqq \frac{t+2s-4}{2} |V(G) \setminus \{x\}| \geqq (s-1)|V(G) \setminus \{x\}|.$$

(If $t=1$, the assertion is trivial.) So $\{S, T=V(G) \setminus S\}$ is a partition of $V(G)$. We shall prove that this is an appropriate partition.

First we remark that $|S| \geqq s+1$. Suppose that $|S|=s$, then from $(*)$ we get

$$e(S) > (s-1)s - \frac{s(s-1)}{2} = \frac{s(s-1)}{2},$$

a contradiction.

Now let x be a point in S with minimum degree in $G(S)$. We claim that $d(x) \geqq s$ i.e. $d(S) \geqq s$. Assume indirectly that it is not the case. As

$$|S \setminus \{x\}| \geqq s,$$

$$e(S \setminus \{x\}) > (s-1)|S| - \frac{s(s-1)}{2} - (s-1) = (s-1)|S \setminus \{x\}| - \frac{s(s-1)}{2}.$$

This contradicts the minimality of $|S|$.

On the other hand, if x is the point above, then by the minimality of $|S|$,

(1)

$$e(S) - d(S) = e(S \setminus \{x\}) \leqq (s-1)|S \setminus \{x\}| - \frac{s(s-1)}{2} = (s-1)(|S|-1) - \frac{s(s-1)}{2}.$$

We have

$$\frac{|S|\, d(S)}{2} \leqq e(S) \leqq (s-1)\left(|S|-1-\frac{s}{2}\right) + d(S), \quad d(S) \leqq (2s-2)\frac{|S|-1-\frac{s}{2}}{|S|-2}.$$

So if $s \geqq 3$, then $d(S) < 2s-2$, i.e. $d(S) \leqq 2s-3$.

Now we show that "few" edges go from any point of $T=V(G)\setminus S$ to S. Let $D=\max_{x\in T} d(x, S)$ and let y be a point of T for which $d(y, S)=D$. We claim that $D\leqq 2s-3$.

There are two cases to consider.

Case 1. $d(S)\leqq 2s-4$.

Suppose that $D>2s-3$. Exchange x for y in S, let S' be the new set. $e(S')>e(S)$, $|S'|=|S|$, so S' satisfies the condition $(*)$. This contradicts the maximality of $e(S)$. This proves that $D\leqq 2s-3$.

Case 2. $d(S)=2s-3$.

If $D\geqq 2s-1$, then exchanging x for y, a contradiction arises again.

If $D=2s-2$ and there is an $x\in S$ such that the degree of x in $G(S)$ is $2s-3$ and $(x, y)\notin E(G)$, then the above exchange leads to contradiction again.

If $D=2s-2$ and there exists no x of this kind, then in S only the neighbours of y may have degree $2s-3$. So

$$e(S)\geqq\frac{(2s-2)(2s-3)+[|S|-(2s-2)](2s-2)}{2}=(s-1)(|S|-1).$$

But from (1)

$$e(S)\leqq(s-1)(|S|-1)-\frac{s(s-1)}{2}+d(S)=(s-1)(|S|-1)-\frac{s(s-1)}{2}+2s-3.$$

So we have

$$2s-3-\frac{s(s-1)}{2}\geqq 0,$$

$$0\geqq s^2-5s+6=(s-2)(s-3).$$

This contradicts $s\geqq 4$.

This proves that $D\leqq 2s-3$, so $d(T)\geqq t$. ∎

In fact we proved the following assertion:

3.2. Theorem. *Let G be a graph with minimum degree at least $2s-1$ $(s\geqq 4)$ and let S be a vertex-set such that $|S|$ is minimal and $e(S)$ maximal under condition $(*)$. Then S, $V(G)\setminus S$ are non-empty sets and in S the degree of any point is at least s and from any point in $V(G)\setminus S$ at most $2s-3$ edges go to S.* ∎

Remarks. 1. Exchanging s for t we obtain $g(s, t)\leqq 2t+s-3$. So if $\min(s, t)\geqq 4$, then $g(s, t)\leqq s+t-3+\min(s, t)$.

2. It can easily be seen from the proof that if $s=3$, then $g(s, t)\leqq t+4$ and if $s=2$, then $g(s, t)\leqq t+3$.

3. The complete graph K_{s+t+1} shows that $g(s, t)\geqq s+t+1$. To sum up our results: If $\min(s, t)\leqq 4$, then $g(s, t)=s+t+1$, if $\min(s, t)\geqq 5$, then $s+t+1\leqq g(s, t)\leqq s+t+1+(\min(s, t)-4)$.

4. In the proof we used only the following conditions on the set S:

(1) S satisfies $(*)$,

(2) if $x \in S$, then $S \setminus \{x\}$ does not satisfy $(*)$,
(3) if $x \in S$ and $y \notin S$, then $e(S \setminus \{x\} \cup \{y\}) \leqq e(S)$.

Based on this observation we give an algorithm which has a graph G with minimum degree at least $2s+t-3$ as input and a partition $\{S, T\}$ as output.

Step 0. Let $S = V(G)$.

Step 1. If there exists an $x \in S$ such that $S \setminus \{x\}$ satisfies $(*)$, then let $S = S \setminus \{x\}$ and start the algorithm again with the new S.

If there is no appropriate x, go to step 2.

Step 2. If there is an $x \in S$ and $y \notin S$ such that $e(S \setminus \{x\} \cup \{y\}) > e(S)$, then let $S = S \setminus \{x\} \cup \{y\}$ and start the algorithm again.

If there is no (x, y) pair as above, then from the above remark $\{S, V(G) \setminus S\}$ is a valid partition.

We do the first type of change at most n times and the second type at most $n^2/2$ times. $(|V(G)| = n.)$ In the first type of change n inspections are needed to check whether there exists such a vertex and in the second type of change $n^2/4$ inspections are needed. Consequently the whole algorithm is finished in no more than $O(n^5)$ steps.

4. Estimating the function $f(s, t)$

For the examination of the connectivity we use the following result of Mader [2]:

4.1. Theorem. (Mader). *If* $|V(G)| \geqq 2n-1$ ($n \geqq 2$, *natural number*) *and* $e(G) > (2n-3)(|V(G)|-(n-1))$ *for a graph* G, *then* G *has an n-connected induced subgraph.* ∎

In fact the proof of this theorem gives the follows result:

4.2. Theorem. (Mader) *If* G *is a graph as above,* $V_0 \subseteqq V(G)$ *and* $|V_0|$ *is minimal such that* $|V_0| \geqq 2n-2$ *and*

$$e(V_0) > (2n-3)(|V_0|-n+1) = (2n-3)|V_0| - \frac{(2n-2)(2n-3)}{2},$$

then G_0 *induces an n-connected subgraph.* ∎

Using this result, we prove the following theorem:

4.3. Theorem. *If* $s \geqq 3$, $t \geqq 2$ *and* G *is an* $(s+t-1)$*-connected graph with* $d(V(G)) \geqq 4s+4t-13$, *there exists a partition* $\{S, T\}$ *of* $V(G)$ *such that* $G(S)$ *and* $G(T)$ *are s-connected and t-connected, respectively.*

Proof. Let $S \subseteqq V(G)$ satisfy the following conditions:

$$|S| \geqq 2s-2,$$

$$e(S) > (2s-3)|S| - \frac{(2s-2)(2s-3)}{2}$$

and $|S|$ be minimal and (if there are more than one such S) $e(S)$ maximal. $\{S, T=V(G)\setminus S\}$ is a partition of $V(G)$ which can be proved in the same way as the proof of preceding theorem.

By Theorem 4.2 S induces an s-connected subgraph.

On the other hand S is the same set as in Theorem 3.1, only $2s-2$ stands instead of s. So by Corollary 3.2, $d(y, S) \leqq 2(2s-2)-3=4s-7$ for $y \in T$. Consequently, $d(T) \geqq 4t-6$. So

$$|T| \geqq 2t-2, \quad \text{and}$$

$$e(T) \geqq \frac{(4t-6)|T|}{2} > (2t-3)[|T|-(t-1)].$$

Then by Mader's theorem 4.2 there exists a subset $T_0 \subseteqq T$ with $G(T_0)$ t-connected.

After this, we can follow the proof of Thomassen. Let S and T be non-empty, disjoint sets such that $G(S)$ and $G(T)$ are s-connected and t-connected, respectively, and $|S \cup T|$ is maximal. (S and T_0 show that such sets exist.)

We shall prove that $S \cup T = V(G)$. Suppose indirectly that $A = V(G) \setminus (S \cup T) \neq \emptyset$. By $|(S \cup A) \cup T| > |S \cup T|$ and the maximality of $|S \cup T|$, $S \cup A$ and T is not a legal partition so $G(S \cup A)$ is not an s-connected graph. Consequently $G(S \cup A)$ has a cut-set B such that $|B| \leqq s-1$. Let G_1 be a component of $G((S \cup A) \setminus B)$ in A and C its vertex-set. $|(T \cup C) \cup S| > |S \cup T|$, so $T \cup C, S$ is not a good pair $G(T \cup C)$ has a cut-set D such that $|D| \leqq t-1$.

It is easy to verify that $B \cup D$ is a cut-set of G. But $|B \cup D| \leqq s+t-2$, contradicting the $s+t-1$-connectivity of G. ∎

Remarks. 1. If G is $(4s+4t-13)$-connected, then G satisfies the conditions of Theorem 4.3. From this we obtain that if $s \geqq 3$, $t \geqq 2$, then $f(s, t) \leqq 4s+4t-13$. **2.** The graph K_{s+t+1} shows that $f(s, t) \geqq s+t+1$.

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